Determinant of a new fermionic action on a lattice. II

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We investigate the fermion determinant of a new action on a $(1+D)$-dimensional lattice for SU(2) gauge groups. This action possesses the discrete chiral symmetry and provides 2^D-component fermions. We also comment on the numerical results on fermion determinants in the $(1+D)$-dimensional SU(3) gauge fields.

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I. INTRODUCTION

As is well known, the lattice formulation of fermions has extra physical particles or breaks chiral symmetry. This is unavoidable under a few plausible assumptions [1]. Several methods have been proposed to deal with this difficulty. Wilson's formulation [2], which is one of the most popular schemes, eliminates the unwanted particles with an additional term which vanishes in the naive continuum limit. However, this formulation sacrifices the chiral symmetry. An alternative scheme is the staggered fermion formulation proposed by Kogut and Susskind [3]. This scheme preserves the discrete chiral symmetry and in this point the staggered fermion has an advantage over the Wilson fermion. But, the staggered formulation describes a theory with $2^{(1+D)/2}$ degenerate quark flavors (2^{1+D} components) in $(1+D)$ dimensions, while there is no restriction on the flavor number in the Wilson formulation. Recently, it has been shown that lattice fermionic actions with the Ginsparg-Wilson relation [4] have an exact chiral symmetry and are free from restriction on the flavor number. But, these actions cannot be ''ultralocal'' [5], which makes numerical simulations complicated.

In recent papers [6,7], we proposed a new type of fermionic action on a $(1+D)$-dimensional lattice. The action is ultralocal and has discrete chiral symmetry. On the Euclidean lattice the minimal number of fermion components is 2^D, which should be compared with 2^{1+D} of the staggered fermion. When dynamical fermions are included, the numerical feasibility relies on the reality and positivity of the fermion determinant. In the previous paper [8] we investigated, analytically and numerically, the fermion determinant of our new action in the $(1+1)$-dimensional U(1) lattice gauge theory. We showed the reality of our fermion determinant under the condition fixing the global phase of link variables along the temporal direction. By a similar discussion to the U(1) gauge group, we could also find the reality and the positivity of our fermion determinant in the $(1+1)$-dimensional SU(N) lattice gauge theory.

In this paper we analytically show that our fermion determinant with the SU(2) gauge fields is real and positive in $(1+D)$ dimensions. We also comment on the numerical results of the fermion determinant in the $(1+D)$-dimensional SU(3) gauge fields, and discuss the effectiveness of our new action for SU(2) and SU(3) lattice gauge theories.

II. NEW FERMIONIC ACTION

In a recent paper [7], we proposed a new fermionic action on the Euclidean lattice. Though the action keeps a discrete chiral symmetry similar to the staggered fermion action, the fermion field has 2^D components in $(1+D)$ dimensions. In this section we briefly sketch our formalism for later convenience.

The action can be written with a fermion matrix Λ as

$$S_f=\sum_{n,m}\psi_n^\dagger\Lambda_{nm}\psi_m, \tag{2.1}$$

where the summation is over lattice points and spinor indices, and our fermion matrix is defined by

$$\Lambda=1-S_0^\dagger U_E. \tag{2.2}$$

Here U_E is the Euclidean time evolution operator and S_μ is the unit shift operator defined by

$$S_\mu\psi(x^0,x^1,\ldots,x^\mu,\ldots,x^D)=\psi(x^0,x^1,\ldots,x^\mu+1,\ldots,x^D)\quad(\mu=0,1,\ldots,D). \tag{2.3}$$

We require that the propagator has no extra poles and find that U_E has the form

$$U_E=1-\sum_{i=1}^{D}\frac{r_E}{2}\{iX_i(S_i-S_i^\dagger)+(1-Y_i)(S_i-2+S_i^\dagger)\}, \tag{2.4}$$

where r_E is the ratio of the temporal lattice constant to the spatial one. The spinor matrices X's and Y's should satisfy the following algebra:

$$\{X_i,X_j\}=\frac{2}{r_E}\delta_{ij},\qquad \{X_i,Y_j\}=0,\qquad \{Y_i,Y_j\}=2\left(\frac{1}{r_E}\delta_{ij}+1\right), \tag{2.5}$$

where i and j run from 1 to D. The matrix $2(\delta_{ij}/r_E+1)$ is positive definite for any positive r_E, therefore X's and Y's can be assumed Hermitian,

$$X_i^\dagger = X_i, \quad Y_i^\dagger = Y_i. \tag{2.6}$$

The matrices X's and Y's can be expressed by the Clifford algebra

$$\Gamma_n^\dagger = \Gamma_n, \quad \{\Gamma_n, \Gamma_m\} = 2\delta_{nm} \quad (n,m = 1, \ldots, 2D) \tag{2.7}$$

in several ways. One is

$$X_i = \sqrt{\frac{1}{r_E}}\Gamma_i, \quad Y_i = \sum_{j=1}^{D} \alpha_{ij}\Gamma_{D+j}, \tag{2.8}$$

as was used in the previous paper. Another one is

$$X_i = \sqrt{\frac{1}{r_E}}\Gamma_i, \quad Y_i = \sqrt{\frac{1}{r_E}}\Gamma_{D+i} + \Gamma_{2D+1}, \tag{2.9}$$

where Γ_{2D+1} is

$$\Gamma_{2D+1} = (-i)^D \Gamma_1 \Gamma_2 \cdots \Gamma_{2D}. \tag{2.10}$$

The latter is more convenient than the former for later use. The dimension of the irreducible representation for Γ's is 2^D and accordingly ψ has 2^D components.

The interaction of the fermion with gauge fields is introduced by replacing the unit shift operators by covariant ones:

$$S_\mu \to S_\mu(x) \equiv U_{x,x+\hat{\mu}} S_\mu, \tag{2.11}$$

where $\hat{\mu}$ is the unit vector along the μ'th direction, and $U_{x,y}$ is a link variable connecting sites x and y. The fermion matrix (2.2) and the time evolution operator (2.4) become

$$\Lambda(x) = 1 - S_0^\dagger(x) U_E(x) \tag{2.12}$$

and

$$U_E(x) = 1 - \sum_{i=1}^{D} \frac{r_E}{2}\{iX_i[S_i(x) - S_i^\dagger(x)] + (1 - Y_i)[S_i(x) - 2 + S_i^\dagger(x)]\}. \tag{2.13}$$

III. FERMION DETERMINANT FOR THE SU(2) CASE

In this section we analytically study the determinant of our fermion matrix in SU(2) gauge fields. First, in the $(1+1)$-dimensional case, the complex conjugation of $U_E(x)$ is

$$U_E^*(x) = 1 - \frac{r_E}{2}\{-iX_1^*[S_1^*(x) + S_1^{*\dagger}(x)] + (1 - Y_1^*)[S_1^*(x) - 2 + S_1^{*\dagger}(x)]\}, \tag{3.1}$$

where we can write

$$S_1(x) = \alpha_0(x)\mathbf{1} + i\sum_{i=1}^{3} \alpha_i(x)\tau_i, \tag{3.2}$$

since the link variables in $S_1(x)$ are SU(2) gauge group elements. Here $\alpha_0(x)$ and $\alpha_i(x)$ are real and depend on lattice points and $\tau_{1,2,3}$ are the Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.3}$$

Then we have

$$S_1(x)\tau_2 = \tau_2 S_1^*(x). \tag{3.4}$$

By the same discussion for other unit shift operators $S_0^\dagger$ and $S_1^\dagger$, we also have

$$S_0^\dagger(x)\tau_2 = \tau_2 S_0^{*\dagger}(x), \quad S_1^\dagger(x)\tau_2 = \tau_2 S_1^{*\dagger}(x). \tag{3.5}$$

If we can find the matrix Γ such that

$$X_1^*\Gamma = -\Gamma X_1, \qquad Y_1^*\Gamma = \Gamma Y_1, \tag{3.6}$$

it is easily shown that

$$(\Gamma \otimes \tau_2)(S_0^\dagger(x) U_E(x))^* (\Gamma \otimes \tau_2) = S_0^\dagger(x) U_E(x). \tag{3.7}$$

For example, we make the following choice:

$$X_1 = \sqrt{\frac{1}{r_E}}\tau_1, \quad Y_1 = \sqrt{\frac{1}{r_E}}\tau_2 + \tau_3, \tag{3.8}$$

the matrix Γ acting on two components fermi fields defined by

$$\Gamma = \tau_3 \tag{3.9}$$

satisfies Eq. (3.6). Equation (3.7) implies that if λ is some eigenvalue of our fermion matrix $\Lambda(x)$, then λ^* is an eigenvalue of $\Lambda^*(x)$ and thus also of $\Lambda(x)$. Therefore eigenvalues of $\Lambda(x)$ are either real or come in complex conjugate pairs. From the above discussion we can prove the reality of our fermion determinant for the SU(2) gauge groups.

Next we show its positivity. We define

$$\Gamma' = (\Gamma \otimes \tau_2)K, \tag{3.10}$$

where K is complex-conjugation operator. We find

$$\Gamma'\Gamma' = (\tau_3 \otimes \tau_2)K(\tau_3 \otimes \tau_2)K = (\tau_3 \otimes \tau_2)[\tau_3 \otimes (-\tau_2)] = -1, \tag{3.11}$$

and from the relation (3.7) we can show

$$[\Gamma', \Lambda(x)] = 0. \tag{3.12}$$

For a real eigenvalue λ_R of $\Lambda(x)$ and the eigenvector v_R for this eigenvalue, from Eq. (3.12) we obtain

$$\Lambda(x)\Gamma' v_R = \Gamma' \Lambda(x) v_R = \Gamma' \lambda_R v_R = \lambda_R \Gamma' v_R. \tag{3.13}$$

Suppose $\Gamma' v_R = c v_R$, then we find

$$\Gamma'^2 v_R = \Gamma' c v_R = c^* \Gamma' v_R = |c|^2 v_R, \tag{3.14}$$

which is inconsistent with Eq. (3.11), so that $\Gamma' v_R$ is different eigenvector for the same eigenvalue. Therefore the eigenvalues on real axis are degenerate in pairs and the determinant of $\Lambda(x)$ is positive.

The above proof of the positivity of our fermion determinant for the SU(2) group can be expanded to higher dimensions. We can make a fundamental representation for the Clifford algebra with $2D$ elements Γ_n $(n=1,\ldots,2D)$ using direct products of the Pauli matrices

$$\Gamma_1 = \tau_1 \otimes \underbrace{\tau_3 \otimes \cdots \otimes \tau_3}_{D-1}$$

$$\Gamma_2 = \mathbf{1} \otimes \tau_1 \otimes \underbrace{\tau_3 \otimes \cdots \otimes \tau_3}_{D-2}$$

$$\vdots$$

$$\Gamma_i = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{i-1} \otimes \tau_1 \otimes \underbrace{\tau_3 \otimes \cdots \otimes \tau_3}_{D-i}$$

$$\vdots$$

$$\Gamma_D = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{D-1} \otimes \tau_1 \tag{3.15}$$

$$\Gamma_{D+1} = \tau_2 \otimes \underbrace{\tau_3 \otimes \cdots \otimes \tau_3}_{D-1}$$

$$\Gamma_{D+2} = \mathbf{1} \otimes \tau_2 \otimes \underbrace{\tau_3 \otimes \cdots \otimes \tau_3}_{D-2}$$

$$\vdots$$

$$\Gamma_{D+i} = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{i-1} \otimes \tau_2 \otimes \underbrace{\tau_3 \otimes \cdots \otimes \tau_3}_{D-i}$$

$$\vdots$$

$$\Gamma_{2D} = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{D-1} \otimes \tau_2$$

where i runs from 1 to D. It can be easily checked that Γ_n's satisfy the relation

$$\{\Gamma_n, \Gamma_m\} = 2\delta_{nm} \quad (n,m=1,\ldots,2D). \tag{3.16}$$

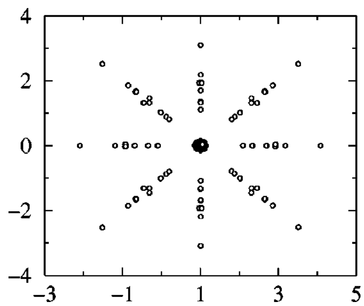

FIG. 1. The spectrum of $\Lambda(x)$ in the complex plane on a $4\times4\times4$ lattice in SU(2) gauge group.

Moreover, we can see that the matrix Γ_i is real and the matrix Γ_{D+i} is pure imaginary:

$$\Gamma_i^* = \Gamma_i, \quad \Gamma_{D+i}^* = -\Gamma_{D+i} \quad (i=1,\ldots,D). \tag{3.17}$$

From the anticommutation relation, we find the Hermite matrix Γ_{2D+1} which anticommutes with all Γ_n's:

$$\Gamma_{2D+1} = (-i)^D \Gamma_1 \Gamma_2 \cdots \Gamma_{2D} = \tau_3 \otimes \tau_3 \otimes \cdots \otimes \tau_3. \tag{3.18}$$

Clearly, the matrix Γ_{2D+1} is Hermitian and the square of this matrix is equal to the unit matrix. Thus, we have

$$\Gamma_{2D+1}\Gamma_i^*\Gamma_{2D+1} = -\Gamma_i,$$
$$\Gamma_{2D+1}\Gamma_{D+i}^*\Gamma_{2D+1} = \Gamma_{D+i}. \tag{3.19}$$

In $(1+D)$ dimensions, the relation Eq. (3.6) is rewritten as follows:

$$X_i^*\Gamma = -\Gamma X_i,$$
$$Y_i^*\Gamma = \Gamma Y_i. \tag{3.20}$$

Then, for the representation of Eq. (2.9),

$$X_i = \sqrt{\frac{1}{r_E}}\Gamma_i, \quad Y_i = \sqrt{\frac{1}{r_E}}\Gamma_{D+i} + \Gamma_{2D+1}, \tag{3.21}$$

we find

$$\Gamma = \Gamma_{2D+1}. \tag{3.22}$$

Since the eigenvalues of $\Lambda(x)$ always consist of complex conjugate pairs and degenerated ones on real axis, we conclude the determinant of our fermion matrix is positive in SU(2) gauge fields in any dimensions.

Now we show a numerical evidence. Figure 1 shows the spectrum of our fermion matrix in a typical background configuration of link variables for SU(2) gauge group in $(1+2)$ dimensions. We find that almost all eigenvalues are distributed on radial lines. As is expected from above discussion, their distribution is symmetric with respect to the real axis and the eigenvalues on real axis are always degenerate. Furthermore, we find that the number of points in Fig. 1 is one half of the number of eigenvalues owing to the degeneracy not only on the real axis but also everywhere. But we

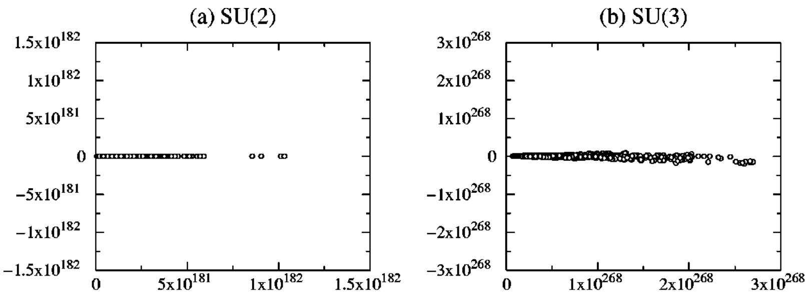

FIG. 2. The distribution in the complex plane of our fermion determinants in (a) SU(2) and (b) SU(3) for each configurations of 2000 Monte Carlo iterations after getting good equilibrium, i.e., after 2000 iterations, on a $4\times4\times4$ lattice at $\beta=2.0$.

have not yet proved this property analytically. Similarly in $(1+3)$ dimensions we can numerically confirm the symmetry with respect to the real axis and the degeneracy of the eigenvalues, and the positivity of the determinant.

IV. DISCUSSION AND SUMMARY

In a previous paper [8] we reported analytical and numerical results on the fermion determinant of our new action in $(1+1)$ dimensions. In the case of U(1) gauge group, we were faced with the problem of convergence in numerical simulations. The cause of the poorness of the convergence is that the summation of the $\det[1-S_0^\dagger(x)U_E(x)]=\det[1-e^{i\Theta}\tilde{S}_0^\dagger(x)U_E(x)]$ over arbitrary phase angle Θ is canceled out accidentally. The element $e^{i\Theta}$ comes from the one sided time difference operator $S_0^\dagger(x)$ with $\theta_0(x)$ replaced by $\theta_0(x)+\Theta$, i.e., $S_0^\dagger(x)=e^{i\Theta}\tilde{S}_0^\dagger(x)$, where $\theta_0(x)$ is defined by $U_{x,x+\hat{0}}=e^{i\theta_0(x)}$ [8]. Therefore we must have control of the phase angle Θ in order to get good convergence in the $(1+1)$-dimensional U(1) gauge theory. In fact we analytically showed that our fermion determinant is real for all configurations and positive for most configurations under the T condition $(\theta_0(x)=0)$, which corresponds to the temporal gauge condition on the infinite lattice, or the GT condition $[\sum_x\theta_0(x)=n\pi:\ n=\text{even}]$, which is achieved by a gauge transformation on the infinite lattice. It was also verified numerically. On the other hand we got good convergence without any conditions in the $(1+1)$-dimensional SU(N) case, because elements such as $e^{i\Theta}$ do not belong to the SU(N) group.

The above discussion is applicable to higher dimensions to a certain extent. In the SU(2) group our fermion determinant is analytically shown real and positive in any dimensions. In Fig. 2(a) we give the numerical evidence that the determinant is real and positive in $(1+2)$ dimensions. In the case of the SU(3) group, we cannot prove the reality of the determinant. But from Fig. 2(b) we see that the distribution of the determinant is concentrated near the real axis without any conditions and the phase angle of the determinant is small. We have obtained similar results in $(1+3)$ dimensions. When the phase angle of the fermion determinant is small enough, we can neglect the phase factor and make use of $|\det\Lambda(x)|$ instead of $\det\Lambda(x)$. In the above numerical simulations link variables are updated by the Metropolis method and determinants are calculated by the LU decomposition. So there are no systematic errors in the determinants. In conclusion, we believe that our new fermionic action is a profitable formulation for the numerical simulations of SU(2) and SU(3) lattice gauge theory.

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